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THE NONLINEAR MEMBRANE SHELL WITH APPLICATION TO NONCIRCULAR CYLINDERS

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Abstract—The nonlinear membrane problem is presented in terms of the curvatures and stresses (or strains) as field variables. As a part of the formulation, appropriate boundary conditions are studied.

The significant features of the equations are discussed. Among the topics are included: dependence on initial shape, types of equations, ability to clarify anomalities of the linear membrane problem, stability and the membrane edge effect problem. A perturbation series scheme for the solution of the equations is also presented.

The usefulness of the theory in solving practical problems is demonstrated by presenting a solution to the problem of the noncircular cylindrical membrane shell under lateral pressure.

INTRODUCTION

THE stress (strain)–curvature formulation of the shell problem has some advantages which make it particularly attractive for nonlinear studies: the simple and intrinsic formulation, symmetry of the equations, the ease of establishing the small strain approximation and the direct geometrical meaning attached to the field variables during the deformation process.

It suffers, though, from an important disadvantage in that it is difficult to assign to it geometrical boundary conditions. One of the objectives of this paper is to present a stress–curvature approach to the nonlinear membrane problem which includes a formulation of appropriate boundary conditions—both geometric and static. It is intended to show that this approach can be made as useful as the displacement approach [1] and should be regarded as complementary to it.

A second objective of this paper is to introduce a perturbation-type solution to the problem of the closed noncircular cylindrical membrane under lateral pressure. It is appropriate to introduce it through a short discussion of the corresponding linear problem and its limitations.

The cylindrical surface is defined by the variable radius of curvature $R(s)$ of the closed plane curve forming its contour. Measured distance along the curve defines the coordinate s. The length of the cylinder along the generator is $2L$ and the corresponding distance coordinate is x.

Let u, v, w be displacements in the x, s and inward normal directions, respectively. Let n^{xx}, n^{ss} and n^{xs} be the membrane stress resultants. The shell thickness is h, E and ν are the elastic constants and p is the normal pressure loading on the shell.

The shell is supported at $x = \pm L$ by frames which are rigid in their plane but offer no out-of-plane resistance. The corresponding boundary conditions are:

$$n^{xx} = 0; \qquad v = 0 \quad \text{at } x = \pm L.$$

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The linear membrane equations for this problem are well known [4, pp. 169–172]. The solution is:

$$n^{ss} = pR \tag{1}$$

$$n^{xs} = -pxR_{,s} \tag{2}$$

$$n^{xx} = \frac{p}{2}(x^2-L^2)R_{,ss}. \tag{3}$$

The expression for the normal displacement is:

$$w = -\frac{pR}{Eh}[\tfrac{5}{6}L^4 - x^2L^2 + \tfrac{1}{6}x^4)R_{,ssss} + (x^2-L^2)R_{,ss} + R] \tag{4}$$

where the comma denotes partial differentiation.

The solution has the following limitations:

1. Experimental evidence and physical reasoning demand that as the length of the noncircular cylinder increases, its shape and state of stress far from the supports will approach those of a circular cylindrical shell. These requirements are not met by the solution, and the results even increase beyond all bounds as L increases.

2. The solution requires the continuity of the derivatives of $R(s)$ up to and including the fourth. It is therefore sensitive to local irregularities which, in a real shell, should be smoothed out by the pressure. The "mathematical" reason is the need for repeated differentiation with respect to s in the solution process, but there is no physical mechanism to support this demand.

3. Boundary conditions along the straight generators cannot be accommodated (important for open shells).

This behavior of the oval shell is well known and corrections have been applied by the introduction of bending effects. Examples are the semi-membrane theory of Vlasov and Novozhilov [4, pp. 254–259] or the oval shell studies [4, pp. 239–254, 9].

The problem will be approached later by the introduction of nonlinear membrane effects. For an order-of-magnitude comparison, the parameter

$$\mu = 12(1-\nu^2)\left(\frac{\sigma}{E}\right)\left(\frac{\lambda}{h}\right)^2 \tag{5}$$

can be taken as a measure of the ratio of nonlinear-membrane to bending corrections in the equilibrium equations. Here σ is a representative membrane stress and λ is a "differentiation-length" of the bending correction. In this respect, linear membrane theory failures may be classified as follows:

(a) *Edge effect failures*

Here $\lambda \sim \sqrt{(Rh)}$ and therefore

$$\mu \sim 12(1-\nu^2)\left(\frac{\sigma}{E}\right)\left(\frac{R}{h}\right). \tag{6}$$

For many practical structures μ would be of order 1 and both effects would be important. A discussion of this case for the axisymmetric torus and circular cylinder is presented, for example, in [3].

(b) *Failure in the large*

Corrections to linear membrane theory are to be applied to large areas of the shell. Here, $\lambda \sim R$ (stability is excluded) and then

$$\mu \sim 12(1-\nu^2)\left(\frac{\sigma}{E}\right)\left(\frac{R}{h}\right)^2. \tag{7}$$

Many if not most aeronautical structures have a large μ and nonlinear membrane effects dominate. The noncircular cylinder problem falls in this class.

THE BASIC EQUATIONS

The use of the metric ($\bar{a}_{\alpha\beta}$) and curvature ($b_{\alpha\beta}$) tensors of a surface (or their increments) as variables for the purpose of formulating the shell problem is well established. This approach, usually in an incremental form, has been used for several shell studies [10–13]. It is adopted in this study but with a difference in that increments are taken in the metric (strains) but not in the curvatures.

For details of the derivations the reader is referred to the above references and to [20]. The resulting equations are as follows:

$$n^{\alpha\beta}|_\alpha + n^{\alpha\theta}(2e^\beta_\alpha|_\theta - e_{\alpha\theta}|^\beta) = 0 \tag{8}$$

$$n^{\alpha\beta}b_{\alpha\beta} = p \tag{9}$$

$$\tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}b_{\alpha\beta}b_{\gamma\delta} + \frac{1}{Eh}\nabla^2 n^\alpha_\alpha = K_0\left(1 - \frac{1-\nu}{Eh}n^\alpha_\alpha\right) \tag{10}$$

$$\varepsilon^{\beta\gamma}[b_{\alpha\beta}|_\gamma - b_{\theta\beta}(e^\theta_\alpha|_\gamma + e^\theta_\gamma|_\alpha - e_{\alpha\gamma}|^\theta)] = 0 \tag{11}$$

where $n^{\alpha\beta}$ is the symmetric stress-resultant tensor, $b_{\alpha\beta}$ is the curvature tensor and p is the normal pressure per unit area. All are related to the deformed surface.

Tensorial operations are performed with respect to the metric $a_{\alpha\beta}$ of the undeformed surface with permutation tensor $\varepsilon^{\alpha\beta}$ and Gaussian curvature K_0. A single bar denotes covariant differentiation with respect to the $a_{\alpha\beta}$ and ∇^2 is the surface Laplacian. The notations are essentially those of [14].

In addition $e_{\alpha\beta}$ denotes the strain tensor defined by

$$e_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}).$$

It is related to the stress tensor by Hooke's law:

$$e_{\alpha\beta} = \frac{1}{Eh}(a_{\alpha\gamma}a_{\beta\delta} - \nu\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta})n^{\gamma\delta}. \tag{12}$$

The basic important assumption made in the derivations is that the strains are small. This is valid for most practical applications.

Equations (8)–(11) serve as the basic system of equations for the nonlinear membrane problem. The unknowns in the equations are the six quantities $n^{\alpha\beta}$ and $b_{\alpha\beta}$ as functions of the convected coordinates u^{α}. The data for the equations are the metric $a_{\alpha\beta}$ of the undeformed state, the loading p, the elastic constants E, ν and the shell thickness h. The equations constitute a sixth order partial differential system.

BOUNDARY CONDITIONS

When a shell problem is formulated in terms of stresses and curvatures, it becomes essential to be able to express physically meaningful boundary conditions (static, geometric or mixed) in terms of these quantities.

The most important type of support which is characterized by stresses (loads) and curvatures is the beam support. It is one of the more common types of supports for shells, but since it is difficult to handle by the displacement formulations—especially in the nonlinear range—it is usually avoided by idealizations.

The beam as a one-dimensional structural entity appears to be a rather natural boundary for a shell which is a two-dimensional structural entity. In this case, the ordinary differential equations of the beam formulated in terms of the static and geometric quantities of the shell at the boundary, become the boundary conditions of the shell. Limiting values of the beam rigidities serve as the rigid support on one hand and the free edge on the other hand.

It is convenient to regard the boundary beam as an oriented curve (strip) in space. The geometry of strips is presented, for example, in [21, chapter 3]. Indeed, letting $\mathbf{u}_1$ and $\mathbf{u}_2$ be unit vectors in the two principal inertia directions of the beam and letting $\mathbf{t}$ be the unit tangent to the boundary curve, such that $\mathbf{u}_2 = \mathbf{t}\times\mathbf{u}_1$, then the material triad thus defined ("principal inertia triad") gives the orientation of the strip. It deforms with the beam and rotates together with the rotation of its cross sections. The following relations then hold [20, 21]:

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa_1\mathbf{u}_1 + \kappa_2\mathbf{u}_2 \tag{13}$$

$$\frac{\mathrm{d}\mathbf{u}_1}{\mathrm{d}s} = -\kappa_1\mathbf{t} - \theta_b\mathbf{u}_2 \tag{14}$$

$$\frac{\mathrm{d}\mathbf{u}_2}{\mathrm{d}s} = \theta_b\mathbf{u}_1 - \kappa_2\mathbf{t}. \tag{15}$$

The quantities κ_1 and κ_2 may be regarded as "normal curvatures" of the curve and θ_b is its "torsion" with respect to the triad. If one adds to these quantities the state of extension of the curve, measured by the strain ε_s along it, then the four quantities $(\kappa_1; \kappa_2; \theta_b; \varepsilon_s)$ as functions of distance s measured along the curve completely determine the state of the strip (beam) during a deformation process.

In order to relate the properties of the beam to the properties of the surface at the boundary, it is convenient to construct along the boundary curve another triad with the following unit vectors:

N unit normal to the surface along the curve;

$\mathbf{u} = \mathbf{t} \times \mathbf{N}$;

t unit vector tangent to the curve.

The differential equations of the "surface oriented" triad are:

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa_n \mathbf{t} - \theta_s \mathbf{u} \tag{16}$$

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa_n \mathbf{N} + \kappa_g \mathbf{u} \tag{17}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \theta_s \mathbf{N} - \kappa_g \mathbf{t} \tag{18}$$

where κ_n, κ_g and θ_s are the normal curvature, geodesic curvature and torsional curvature of the surface along the boundary curve, respectively.

Let the surface oriented triad make an angle β with the material triad such that $\mathbf{N} \cdot \mathbf{u}_1 = \cos\beta$. Then [21, p. 74]

$$\kappa_1 = \kappa_n \cos\beta + \kappa_g \sin\beta \tag{19}$$

$$\kappa_2 = -\kappa_n \sin\beta + \kappa_g \cos\beta \tag{20}$$

$$\theta_s = \frac{\mathrm{d}\beta}{\mathrm{d}s} + \theta_b. \tag{21}$$

It is possible to relate the properties of the surface-oriented curve to the metric $\bar{a}_{\alpha\beta}$ and curvatures $b_{\alpha\beta}$ of the surface at the boundary. Taking the boundary as the parametric line $u_2 = c$ of the surface (similar relations exist for $u_1 = c$). Then:

$$\kappa_n = (\bar{a}_{11})^{-1} b_{11} \approx a_{11}^{-1} b_{11}(1 - 2\varepsilon_s) \tag{22}$$

$$\kappa_g = (\bar{a})^{\frac{1}{2}}(\bar{a}_{11})^{-\frac{1}{2}}\Gamma_{11}^2 \approx a^{\frac{1}{2}} a_{11}^{-3/2}[\Gamma_{11}^2(1 + \mathrm{e}_\alpha^\alpha - 3\varepsilon_s) + 2\,\mathrm{e}_1^2|_1 - e_{11}|^2] \tag{23}$$

$$\theta_s = (\bar{a})^{\frac{1}{2}}(\bar{a}_{11})^{-1} b_1^2 \approx a^{\frac{1}{2}} a_{11}^{-1} b_1^2(1 + e_\alpha^\alpha - 2\varepsilon_s) \tag{24}$$

$$\varepsilon_s = (a_{11})^{-1} e_{11}. \tag{25}$$

Here $a_{\alpha\beta}$; $a = \det(a_{\alpha\beta})$ and the Christoffel symbols $\Gamma_{\beta\gamma}^\alpha$ are related to the undeformed state. All other quantities belong to the deformed state. The equations are linearized in the strains $e_{\alpha\beta}$.

The first two expressions are well known from differential geometry [16, pp. 80, 130]. The third is obtained from the Weingarten equations [14, equation (1.13.47)] and the definitions of $\mathbf{u}$ and θ_s. The fourth is the obvious relationship between the differential of length along the curve and the surface metric.

The equilibrium equations of the boundary beam are constructed with respect to the principal inertia system. Their derivation is standard and will not be repeated here (for further details see [20]). The final form is:

$$\frac{\mathrm{d}F}{\mathrm{d}s}+\kappa_1\left(\frac{\mathrm{d}M_1}{\mathrm{d}s}-\theta_b M_2\right)-\kappa_2\left(\frac{\mathrm{d}M_2}{\mathrm{d}s}+\theta_b M_1\right) = -(n_T+\kappa_n m_T)+p_T \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}M_1}{\mathrm{d}s}-\theta_b M_2+\kappa_2 M_T\right)-\theta_b\left(\frac{\mathrm{d}M_2}{\mathrm{d}s}+\theta_b M_1+\kappa_1 M_T\right)-\kappa_1 F$$
$$= \left(q_n-\frac{\mathrm{d}m_T}{\mathrm{d}s}\right)\cos\beta+(n_n+\theta_s m_T)\sin\beta+p_1 \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}M_2}{\mathrm{d}s}+\theta_b M_1+\kappa_1 M_T\right)+\theta_b\left(\frac{\mathrm{d}M_1}{\mathrm{d}s}-\theta_b M_2+\kappa_2 M_T\right)+\kappa_2 F$$
$$= \left(q_n-\frac{\mathrm{d}m_T}{\mathrm{d}s}\right)\sin\beta-(n_n+\theta_s m_T)\cos\beta-p_2 \tag{28}$$

$$\frac{\mathrm{d}M_T}{\mathrm{d}s}-\kappa_2 M_1-\kappa_1 M_2 = -m_n+p_m \tag{29}$$

where

- F is the normal force in the beam;
- M_1, M_2 are the bending moments in the beam;
- n_T, n_n are the shell membrane stress resultants at the boundary, and in the directions parallel and normal to it;
- m_T, m_n are the shell twisting and bending moments at the boundary;
- q_n is the transverse shear of the shell at the boundary;
- p_T, p_1, p_2, p_m are additional external loads and moments on the beam in the corresponding directions (if any).

Assuming now that the beam behaves according to strength-of-materials theory, the beam static quantities may be expressed in terms of its geometric increments as follows:

$$M_1 = EI_1\,\Delta\kappa_1 \tag{30}$$

$$M_2 = EI_2\,\Delta\kappa_2 \tag{31}$$

$$F = EA\varepsilon_s \tag{32}$$

$$M_T = GJ\,\Delta\theta_b \tag{33}$$

where EI_1, EI_2, EA, GJ are the beam bending, extensional and torsional rigidities respectively. The symbol Δ denotes the difference in a given quantity between the deformed and the initial states.

Substitution of (30)–(33) and (19)–(21) into (26)–(29) yields four differential equations in the shell quantities $(n_n, n_T, q_n, m_n, m_T, \kappa_n, \kappa_g, \varepsilon_s, \theta_s)$ and the auxiliary parameter β. These equations constitute the boundary conditions for the most general beam-type support.

The development so far did not differentiate between bending and membrane theories. The distinction can be made now since classical bending theory requires rotational continuity between the shell and its boundary beam, leading to the additional boundary equation: $\Delta\beta = 0$. Nonlinear membrane theory, on the other hand, imposes only positional but no rotational continuity between the shell and its boundary beam. In such a case β is unrestricted and its elimination (in principle) from the four equations (26)–(29) leaves only three effective boundary conditions for imposition on the shell (as against four in the bending case). In addition, the bending quantities m_n, m_T and q_n are henceforth taken out of the equations. Substantial simplification of the boundary equations (26)–(33) can be obtained for special cases or if further assumptions are made. A detailed discussion is given in [20] where such topics as rigid support, free edge, linearization, weak curvatures, adaptive edge beams and non-beam-type conditions are considered. Alternatively, one may follow the approach of the example given in this paper and reduce the equations by using the properties of the particular problem at hand. This second approach possesses the important advantage of understanding the physical meaning and various possibilities at each step.

THE PERTURBATION SERIES APPROACH

A convenient approach to studying the equations and to obtaining approximate solutions of the nonlinear system presented above is that of a perturbation series about a known solution (see also [1]). A nonlinear partial differential system can be classified (e.g. elliptic, parabolic, hyperbolic) only with respect to a given solution and in its neighbourhood. The first term in the perturbation scheme can therefore be used to study the behaviour of the nonlinear membrane problem in the neighbourhood of any point in the solution domain. Such a point (which need not be the undeformed state) is specified by the values of the curvature $b_{\alpha\beta 0}$ and stress $n_0^{\alpha\beta}$ tensors. Let ε be any single quantity which can describe the variation of the solution about the point. A perturbation scheme may be started from the known point by expanding the variables in series in ε. Thus:

$$n^{\alpha\beta} = n_0^{\alpha\beta} + \sum_1^{\infty} n_m^{\alpha\beta}\varepsilon^m \tag{34}$$

$$b_{\alpha\beta} = b_{\alpha\beta 0} + \sum_1^{\infty} b_{\alpha\beta m}\varepsilon^m \tag{35}$$

and so forth. Here and in the following, the naught index refers to the known solution.

Substituting into (8)–(11) and separating equal powers in ε, sets of linear differential equations and boundary conditions are obtained, with the "naught" solution functions appearing as coefficients in the equations. (For details, see [20].) Higher perturbation equations can be easily obtained too. It is observed that their homogeneous parts are exactly the same as those of the first perturbation with $n_m^{\alpha\beta}$ and $b_{\alpha\beta m}$ replacing $n_1^{\alpha\beta}$ and $b_{\alpha\beta 1}$ as variables.

For simplicity, the linearized system will be examined specifically for the class of cylindrical surfaces. It will be assumed that the point of expansion is the cylindrical surface of the introduction under a constant state of stress. This can be realized, for example, by applying constant edge loads to the shell. Based on the linearized equations one may use a stress function defined by (see example for a more refined definition):

$$n_1^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\phi_{,\gamma\delta}.$$

Substitution and use of an elimination process lead to an equation of the form:

$$b_{ss0}^2\phi,_{xxxx}-\frac{1}{Eh}n_0^{ij}\nabla^4\phi,_{ij}+\frac{1}{Eh}[\text{lower order terms}] = b_{ss0}p_{1,xx}. \tag{36}$$

For more details, see example or [20]. A similar elimination process for the general shell is complicated by the variable metric and stress fields. However, that in so far as the leading terms (highest order derivatives) are concerned, the features of the process are preserved. In order to establish the nature of the equations, the following partial differential operators are defined:

$$\nabla^4F = F|_{\alpha\beta}^{\alpha\beta} = \text{surface biharmonic operator}$$

$$L_bF = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}b_{\alpha\beta0}F|_{\gamma\delta} = \text{curvature operator}$$

$$L_nF = \frac{1}{Eh}n_0^{\alpha\beta}F|_{\alpha\beta} = \text{stress operator}.$$

The equation of the first perturbation for the general membrane would then read (in principle):

$$(L_b^2-L_n\nabla^4)\phi+[\text{lower derivatives terms}] = L_bp_1.$$

DISCUSSION

(a) *Nature of the equations*

The behaviour of the nonlinear membrane problem near a given solution is governed by the three partial differential operators $L_n\nabla^2\nabla^2$. The system is therefore always twice elliptic, because of the biharmonic operator, but the stress operator L_n may be elliptic, parabolic or hyperbolic, depending on whether $\det(n_0^{\alpha\beta})$ is greater than, equal to or smaller than zero. The situation is essentially different from linear membrane theory where the nature of the equations depends heavily on the undeformed curvatures.

As a result, the number of boundary conditions on each boundary of the shell and in the neighborhood of a given solution depends on $\det(n_0^{\alpha\beta})$. At least two may be specified around the shell. If $\det(n_0^{\alpha\beta}) > 0$ then a third condition should also be specified around the shell. Otherwise the arrangement of the third condition should depend on the properties of the region and initial stress field, as discussed in text books on partial differential equations [17]. For example, a state of biaxial tension is elliptic, pure shear is hyperbolic, whereas a uniaxial stress field is parabolic.

For most structures and materials, the inequality $n_0^{ij} \ll Eh$ holds. The operators $L_n\nabla^2\nabla^2$ have consequently a very small coefficient. This implies an increase in the order and nature of a partial differential system (linear membrane theory about the initial shape) through the addition of a higher order partial differential operator with small coefficients (nonlinear effects). This situation, which is similar to the addition of the bending effects to linear membrane theory, has been noted before and used to obtain solutions of discontinuity and edge effect problems within the framework of the membrane state of stress, mostly for axisymmetric problems of shells of revolution [3, 6, 8, 5, 1 and others]. The relative merits of the stress operator as against the bending operator, has been discussed, among others, by Jordan [3]. It is noted that even though the membrane operator is of lower order than the bending operator, its coefficients are substantially larger for usual thin walled structures (increasing approximately as R/h) and this should make it an important contributor to any correction to linear membrane theory.

In addition to its use as "boundary fixer", nonlinear membrane theory has important uses when the linear theory fails over large regions of the shell. Such is the case when substantial deformations occur which necessitate consideration of their effect on equilibrium, when an inextensional deformation field is superposed on a loaded shell, etc. Some of the effects and contributions of the nonlinear membrane state of stress are shown in the example.

(b) *Classes of shells*

It is observed that the curvatures of the unloaded shell do not appear in the formulation of the nonlinear membrane problem. This implies that any group of surfaces which are geometrically applicable to each other have identical equations for their membrane behaviour. Such a class is, for example, that of developable surfaces and, in particular, cylindrical surfaces.

When a solution to a membrane problem is sought in terms of the perturbation series, the seemingly obvious choice for expansion is the unloaded, underformed shell. This point, though, is usually not acceptable if boundary corrections are to be applied. The reason is that the unloaded shell is a degenerate point in the solution domain since the stress-operator vanishes there and the order of the equations reduces then from sixth to fourth. In addition, in some cases, the expansion may be of the wrong type for the boundary conditions at hand (the type depends in such a case on the sign of the Gaussian curvature K_0 of the underformed surface) and may also be sensitive to discontinuities in the data (see example). The next choice is to take a loaded shell acting on an approximately underformed geometry ($n_0^{\alpha\beta} \neq 0$; $b_{\alpha\beta} = b_{\alpha\beta 0}$). This may be acceptable if the corresponding membrane problem is well posed. The $n_0^{\alpha\beta}$ may be taken from the linear solution. This choice loses its advantages when the expected deformations are not small, when the boundary is deforming too, when the nature of the differential system changes during the deformation process and in cases when the linear problem cannot be solved. Other surface configurations within the class may then be chosen, depending on expected behaviour, convergence considerations and convenience (see cylindrical shell example). In some cases of ill posed problems the perturbation series approach may even fail and resort would have to be made to singular perturbation techniques. It is suggested that some failures of expansion about an unloaded state with the pressure as a parameter or of stability analyses may be of this nature.

(c) *Uniqueness and stability*

The uniqueness of the perturbation expansion about a point ($n_0^{\alpha\beta}$, $b_{\alpha\beta 0}$) depends on the behaviour of the homogeneous part of the first perturbation equations (with corresponding homogeneous boundary conditions). A point for which nontrivial solutions to be equations do not exist, is-stable in the membrane sense. A set of stable points is a stable region in the solution domain. It is suggested that this type of stability is tied to the usual shell stability problem: when a point is unstable in the membrane sense, then it will buckle at some finite shell thickness. The buckling thickness can be predicted by bending theory only, but the determination of the stability regions is a membrane problem.

It has been stated that the criterion for the stability regions is the occurrence of a negative principal stress-resultant. This is not obvious from the perturbation equation. A case at hand is the cylindrical shell where tension in one direction influences the critical load in the second direction. Hence, it is necessary to establish for each particular case the boundaries of the stability region. The points of perturbation expansions can be chosen only from stable regions.

A point may be raised here as to the relative roles of the membrane and bending effects in shell stability analysis. If a stability analysis is attempted solely with the membrane part of the shell equations, it is found that a solution is impossible, as analysis of the stability equations for the cylindrical shell shows. What happens is that the critical load is mode-dependent, decreasing in value with the increase in the number of waves and approaching zero as the wave index m goes to infinity. A mechanism which acts in the opposite direction—e.g. being able to absorb more loads as the number of waves increases—should therefore be added as a necessary part of the stability equations. This mechanism is provided by the bending terms which react in proportion to the changes-in-curvature and therefore increase their resistance as m increases. The balance (critical load) is reached when the bending effects become just sufficient to change the downward trend of the p–m curve of the membrane part. This cannot occur, however, before the "differentiation length" of the waves λ (see Introduction) is such that the bending effects reach the same order of magnitude as membrane effects. But since in a very thin shell the coefficients of the bending effects are much smaller than those of the membrane effect, it follows that the characteristic wave length will be much smaller than the surface characteristic lengths. Therefore the bending effect is in such cases mostly a local effect which is not significantly influenced by overall shell geometry or boundary conditions.

One may thus view the shell stability problem as a nonlinear membrane problem with an additional local mode-sensitive mechanism to offset the downward trend of the p–m curve. It is suggested that the difference between one shell and the other, one boundary condition and the other for the thin shell is expressed mostly in the membrane part and much less so in the bending part. This behaviour agrees with studies of the stability of thin cylindrical shells which show more sensitivity to variations in the membrane boundary conditions than to variations in the bending boundary conditions [18].

EXAMPLE: THE NONCIRCULAR CYLINDRICAL SHELL UNDER LATERAL PRESSURE

The geometry and properties of the shell are given in the introduction. The support conditions of the shell at $x = \pm L$ are given by the following boundary ring properties:

$$EA = EI_1 = \infty; \qquad EI_2 = GJ = 0; \qquad \beta_0 = 0.$$

The case represents rings which are stiff in their plane (which is normal to the shell) but have negligible torsional and out-of-plane stiffnesses. Expression in terms of boundary-beam variables [equations (30)–(33)] yields:

$$\kappa_1 = 1/R(s); M_2 = M_T = \varepsilon_s = 0. \tag{37}$$

The following notations are now introduced:

$$S = \text{perimeter of the underformed contour};$$

$$R_0 = S/2\pi = \text{radius of the isoperimetric circle};$$

$$\xi = x/L \ (-1 \le \xi \le 1);$$

$$\eta = s/R_0 \ (0 \le \eta \le 2\pi);$$

$$[\kappa_x, \kappa_t, \kappa_s] = R_0[b_{xx}, b_{xs}, b_{ss}];$$

$$[N_x, N_t, N_s] = 1/Eh[n^{xx}, n^{xs}, n^{ss}];$$

$$\alpha = R_0/L;$$

$$\bar{M} = M_1/EhR_0^2;$$

$$\bar{p} = pR_0/Eh;$$

$$\rho = (\kappa_s)^{-1}.$$

The use of ρ as a variable offers some advantages over the more common use of κ_s. Noting that the x–s system is Cartesian, equations (8)–(12) become:

$$\alpha N_{x,\xi} + N_{t,\eta} + \alpha N_x(N_x - \nu N_s),_\xi + 2N_t(N_x - \nu N_s),_\eta + N_s[2(1+\nu)N_{t,\eta} - \alpha(N_s - \nu N_x),_\xi] = 0 \quad (38)$$

$$\alpha N_{t,\xi} + N_{s,\eta} + N_s(N_s - \nu N_x),_\eta + 2\alpha N_t(N_s - \nu N_x),_\xi + N_x[2(1+\nu)\alpha N_{t,\xi} - (N_x - \nu N_s),_\eta] = 0 \quad (39)$$

$$\rho N_x \kappa_x + 2\rho N_t \kappa_t + N_s = \bar{p}\rho \quad (40)$$

$$\rho\kappa_{x,\eta} - \alpha\rho\kappa_{t,\xi} - \rho\kappa_x(N_x - \nu N_s),_\eta - \alpha\rho\kappa_t(1+\nu)(N_s - N_x),_\xi + 2\alpha(1+\nu)N_{t,\xi} - (N_x - \nu N_s),_\eta = 0 \quad (41)$$

$$\rho^2\kappa_{t,\eta} + \alpha\rho,_\xi - \rho^2\kappa_x[2(1+\nu)N_{t,\eta} - \alpha(N_s - \nu N_x),_\xi] - \rho^2\kappa_t(1+\nu)(N_s - N_x),_\eta + \alpha\rho(N_s - \nu N_x),_\xi = 0 \quad (42)$$

$$\kappa_x - \rho\kappa_t^2 + \alpha^2\rho(N_x + N_s),_{\xi\xi} + \rho(N_x + N_s),_{\eta\eta} = 0. \quad (43)$$

(a) *Boundary conditions*

Substitution of (37) into (29) gives

$$\kappa_2 M_1 = 0.$$

Excluding special cases (e.g. axisymmetric problem), the branch $M_1 = 0$ is not accepted because of the infinite rigidity in the $\mathbf{u}_1$ direction. Therefore $\kappa_2 = 0$.

In cases where a strong boundary-layer does not exist near the supports, the equations simplify further since κ_g which involves only strain derivatives must be small. It follows from

(19) and (20) that β must be small too and the boundary conditions reduce after elimination to:

$$\rho = \frac{R(\eta)}{R_0} \tag{44}$$

$$\varepsilon_s = 0 \tag{45}$$

$$\rho[\rho(\bar{M},_{\eta\eta} - \kappa_t^2 \bar{M})],_\eta + \bar{M},_\eta = -\rho N_t \tag{46}$$

$$(\kappa_t \bar{M}),_\eta + \kappa_t \bar{M},_\eta = -N_x \tag{47}$$

with $\bar{M}$ acting as an auxiliary parameter. The number of effective conditions is three.

Of all the conditions, (45) is apt to cause edge-effect-type disturbances. This has been demonstrated [5, 19] for the axisymmetric case. Indeed, in this case one obtains from the general equations:

$$N_s - \nu N_x = 0; \qquad N_t = 0; \qquad N_x \cos\beta = 0.$$

The case $N_x = 0$ is now unacceptable since it contradicts (40) and therefore $\cos\beta = 0$ or $\beta = \pi/2$. But now κ_g must be large, implying large strain rates at the boundary which are typical of the membrane edge effect case.

The nonaxisymmetric problem is more complicated but leads to essentially the same situation. The corresponding physical problem of a pressurized cylindrical membrane which is prevented from expanding at its boundary is well known.

(b) *The perturbation series*

In order to obtain an approximate solution to the nonlinear shell problem defined above, a solution is sought in terms of a perturbation series about a convenient point in the solution domain.

Expansion about the underformed state is ruled out since it produces linear membrane theory as its first perturbation, with all the disadvantages listed in the Introduction. Attention is therefore focused on the second limiting point in the solution domain of the equations—that of the circular cylinder. This point has the advantage of being approached by all noncircular cylinders as the pressure or the length increase. In addition, the resulting perturbation equations are with constant coefficients.

The process of expansion about the circular shape has a direct physical meaning: given a closed cylindrical sheet with perimeter S but with an unspecified contour, one may "produce" the loaded shell by two distinct methods: (a) first form the contour and then load the shell; (b) first load the shell and then form the contour. The first approach is represented by expanding around the original shape while the second is represented by expanding around the circular shape.

The perturbation equations about the circular shape are in this case once parabolic. Hence only two out of the three boundary conditions can be satisfied. The condition on ε_s is therefore dropped. This is not considered to be very important since the ε_s condition results, as noted before, in a "boundary layer" type correction that does not influence the behaviour of the shell in the large, (which is the more important aspect of the example). Since the omission prevents the development of edge-effect-type disturbances, the simplified boundary equations (44), (46), (47) can be used for the analysis.

The solution point of the circular cylinder is expressed in terms of the variables as follows:

$$[N_s = \bar{p};\quad \rho = 1;\qquad N_x = N_t = \kappa_x = \kappa_t = \bar{M} = 0].$$

All the variables of the problem are now expanded in perturbation series about this point. The perturbation parameter ε is selected as some measure of the deviation from circularity of $R(\eta)$.

Thus

$$N_s = \bar{p} + \sum_{m=1}^{\infty} N_{sm}\varepsilon^m$$

$$N_x = \sum_{1}^{\infty} N_{xm}\varepsilon^m$$

and so forth.

It is assumed now that the underformed contour has a continuously turning tangent and that its radius of curvature $R(\eta)$ is sectionally continuous and bounded in $0 < \eta < 2\pi$. Only symmetric curves are considered for simplicity. Hence $R(\eta)$ may be expanded as a Fourier cosine series in η:

$$R(\eta) = \bar{R}_0 + \sum_{n=2}^{\infty} R_n \cos n\eta. \tag{48}$$

The term with $n = 1$ is excluded since it does not correspond to a closed curve ([20, Appendix B]). The quantity

$$\varepsilon = [R_0^{-1}(\bar{R}_0 - R_0)]^{\frac{1}{2}} \tag{49}$$

is chosen as a measure of noncircularity. It can be shown ([20, Appendix B]) that $\bar{R}_0 \geq R_0$, equality being obtained for the circular cylinder only. Hence it is a proper measure. Thus:

$$R(\eta) = R_0\left[1 + \varepsilon\left(\sum_{2}^{\infty} \lambda_n \cos n\eta\right) + \varepsilon^2\right] \tag{50}$$

where

$$\lambda_n = R_n(\varepsilon R_0)^{-1}. \tag{51}$$

Substitution into the differential equations and boundary conditions, yields the following equations for the first perturbation (linear terms in ε):

$$\alpha N_{x1,\xi} + N_{t1,\eta} + \bar{p}[2(1+\nu)N_{t1,\eta} - \alpha(N_{s1} - \nu N_{x1}),_{\xi}] = 0 \tag{52}$$

$$\alpha N_{t1,\xi} + N_{s1,\eta} + \bar{p}(N_{s1} - \nu N_{x1}),_{\eta} = 0 \tag{53}$$

$$N_{s1} = \bar{p}\rho_1 \tag{54}$$

$$\kappa_{x1,\eta} - \alpha\kappa_{t1,\xi} + 2\alpha(1+\nu)N_{t1,\varepsilon} - (N_{x1} - \nu N_{s1}),_{\eta} = 0 \tag{55}$$

$$\kappa_{t1,\eta} + \alpha\rho_{1,\xi} + \alpha(N_{s1} - \nu N_{x1}),_{\varepsilon} = 0 \tag{56}$$

$$\kappa_{x1} + \alpha^2(N_{x1} + N_{s1}),_{\xi\xi} + (N_{x1} + N_{s1}),_{\eta\eta} = 0 \tag{57}$$

and at $\xi = \pm 1$.

$$\rho_1 = \sum_2^\infty \lambda_n \cos n\eta$$

$$N_{x1} = 0$$

$$\bar{M}_{1,\eta\eta\eta} + \bar{M}_{1,\eta} + N_t = 0.$$

The last equation is now redundant and is to be used for calculating $\bar{M}_1$ after the equations are solved. It should be mentioned here that the equations are similar to the membrane part of the stability equations of the circular cylinder (compare, for example with [12]) except that ρ_1, is used as a variable instead of κ_{s1}.

(c) *Solution of the equations*

The small strain approximation leads to the inequality $\bar{p} \ll 1$. In this case equations (52) and (53) are satisfied by a stress function as follows:

$$N_{x1} = \phi,_{\eta\eta} + \bar{p}\alpha^2\phi,_{\xi\xi} \tag{58}$$

$$N_{s1} = \alpha^2\phi,_{\xi\xi} + \nu\bar{p}\phi,_{\eta\eta} \tag{59}$$

$$N_{t1} = -\alpha\phi,_{\xi\eta}. \tag{60}$$

Substitution into the other equations and repeated use of $\bar{p} \ll 1$ lead to:

$$-\frac{\alpha^4}{\bar{p}}\phi,_{\xi\xi\xi\xi} + [(2+\nu)\alpha^2\phi,_{\xi\xi} + \phi,_{\eta\eta} + \nabla^4\phi],_{\eta\eta} = 0 \tag{61}$$

with boundary conditions at $\xi = \pm 1$:

$$\phi,_{\xi\xi} = \frac{\bar{p}}{\alpha^2}\sum \lambda_n \cos n\eta$$

$$\phi,_{\eta\eta} = 0.$$

The function ϕ is now developed as a Fourier cosine series in η:

$$\phi = \sum_{n=2}^\infty \phi_n \cos n\eta.$$

The equation for ϕ_n becomes (the ν terms have been dropped since they can be shown to have little influence on the results [20]):

$$\frac{1}{\theta^4}\phi_{n,\xi\xi\xi\xi} - 2\left(\frac{n}{\alpha}\right)^2 \phi_{n,\xi\xi} + \left(\frac{n}{\alpha}\right)^4 \phi_n = 0 \tag{62}$$

where

$$\theta = \left[\frac{\bar{p}(n^2-1)}{1+\bar{p}(n^2-1)}\right]^{\frac{1}{4}}. \tag{63}$$

The solution of the differential equation which satisfies the boundary condition is:

$$\phi_n = \frac{\bar{p}\lambda_n}{\alpha^2}\mu_n(\xi) \tag{64}$$

where

$$\mu_n(\xi) = \frac{\cosh\theta_1\cos\theta_2\sinh\theta_1\xi\sin\theta_2\xi - \sinh\theta_1\sin\theta_2\cosh\theta_1\xi\cos\theta_2\xi}{2\theta_1\theta_2(\cosh^2\theta_1 - \sin^2\theta_2)} \tag{65}$$

$$\theta_{1,2} = \frac{n\theta}{2\alpha}(2\pm 2\theta^2)^{\frac{1}{2}}. \tag{66}$$

A study of the results shows that for $\bar{p} \ll 1$ the contributions of the terms $\bar{p}\alpha^2\phi,_{\xi\xi}$ and $v\bar{p}\phi,_{\eta\eta}$ to the stresses N_{x1} and N_{s1} are negligible. Thus the $\bar{p}$ terms in (52) and (53) may be dropped. The same cannot be said about the extra terms in the Codazzi equations (55) and (56) (the neglecting of which leads to the well known "shallowness approximation") since they modify n^2 to become (n^2-1) and this effect can be neglected for large n only [12]. The apparent ability to omit the $\bar{p}$ terms in the equilibrium equations is not a general property. The inability of the Donnell equations to predict the column buckling of cylindrical shells is an example of such oversimplification.

Before expressions for the stress resultants are written and in order to improve accuracy, an artifice is introduced:

The closed cylindrical surface, being multiply-connected, requires Césaro-type integral conditions to assure that it remains closed after deformation (see Appendix B of [20] for more details). The requirement for angular compatibility is expressed by:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\eta}{\rho(\eta)} = 1.$$

Introducing the series expansion for ρ and integrating, one obtains:

$$\rho_0 = 1 + \tfrac{1}{2}\varepsilon^2\sum(\lambda_n\mu_n'')^2 + \dots \tag{67}$$

or, to the same degree of approximation:

$$\rho_0 = \frac{1}{R_0}\left[\bar{R}_0^2 - \sum R_n^2[1-(\mu_n'')^2]\right]^{\frac{1}{2}}. \tag{68}$$

It is noted that the use of $\rho_0 = 1$, as before, satisfies (67) to within linear terms in ε. It is suggested, though, that the use of (67) or (68) in the expressions for n^{ss} and for ρ gives an improved (second order in ε) estimate. The new ρ_0 does not have to satisfy the equations of the first perturbation, but is to be introduced as an imput into the second perturbation. The special form of (40) which has a linear term in ρ assures that this imput will be of the same order in ε as other imputs from the first perturbation. The process may be repeated by imposing (67) on the second perturbation and introducing it as an input into the third, and so on. It thus seems to be a consistent procedure.

The expressions for the physical membrane stress resultants now become:

$$n^{xx} = Eh\varepsilon\phi,_{\eta\eta} = -\frac{p}{\alpha^2}\sum_2^{\infty} n^2\mu_n(\xi)R_n \cos n\eta \tag{69}$$

$$n^{xs} = -Eh\varepsilon\alpha\phi,_{\eta\xi} = \frac{p}{\alpha}\sum_2^{\infty} n\mu_n'(\xi)R_n \sin n\eta \tag{70}$$

$$n^{ss} = p\left\{\left[\bar{R}_0^2 - \sum R_n^2[1-\mu_n''(\xi)^2]\right]^{\frac{1}{2}} + \sum_2^{\infty} \mu_n''(\xi)R_n \cos n\eta\right\}. \tag{71}$$

(d) *Some properties of the solution*

Examination of the solution (64) and of the expressions for the stress resultants (69)–(71) shows some important features which are summarized below:

(a) The membrane stress resultants approach the results of linear membrane theory as the pressure decreases or as the shell becomes shorter.

(b) The geometry of the shell approaches that of the original unloaded shell as the pressure goes to zero.

(c) For light loads, the geometry tends to the results of linear membrane theory in the sense that the same tangential displacements are approached, but the normal displacement is approached up to linear terms in the deviations from circularity.

(d) The solution series provides convergent and continuous results even if $R(\eta)$ or its derivatives have a finite number of discontinuities around the circumference. It smooths irregularities in the shape of the undeformed shell.

(e) If, for a given shell problem, $R(\eta)$ is regular and $\gamma \ll 1$, where

$$\gamma^4 = \bar{p}/\alpha^4 = \frac{pL^4}{EhR_0^3} \tag{72}$$

then the shell may be analyzed by linear membrane theory. If either of these conditions is not met, the solution derived here should be used in full.

(f) The deviations from circularity tend to decrease as the load increases or as the shell becomes longer. For long shells, the solution approaches that of a circular cylinder with radius R_0.

(g) As the harmonic index increases, the equation for the stress function approaches the biharmonic equation of plane elasticity. The latter may be said to be satisfied "in the small" on the surface (in contrast with linear membrane theory which cannot satisfy it).

It appears, that the solution presented here overcomes the difficulties of linear membrane theory but is still consistent with it for lightly loaded, not too long, regular shells.

(e) *The curvature formulation*

This formulation is more adaptable to cases with angular discontinuities in the circumference. Its main disadvantage is that it needs more terms in the perturbation series if the shell is strongly noncircular.

The procedure involves introduction of $\kappa_s = (\rho)^{-1}$ into equations (40)–(43) and expansion in a perturbation series as before. On the boundary, $1/R(\eta)$ is expended as a Fourier cosine series:

$$\frac{R_0}{R(\eta)} = 1 + R_0 \sum_2^\infty k_n \cos n\eta = 1 + \varepsilon \sum_2^\infty \nu_n \cos n\eta$$

where k_n or ν_n are the coefficients of the expansion.

Comparison of the resulting equations and boundary conditions of the first perturbation with those of the radius-of-curvature formulation shows that the two become identical if the following substitutions are made:

$$\kappa_{s1} \text{ replaces } (-\rho_1); \qquad \nu_n \text{ replaces } (-\lambda_n)$$

$$R_0^2 k_n \text{ replaces } (-R_n); \quad R_0 \text{ replaces } [\bar{R}_0^2 - \sum R_n^2[1-\mu_n''(\xi)^2]]^{\frac{1}{2}}$$

with these changes all previous results of the first perturbation solution may be used.

For use in cases with angular discontinuities, let $\chi(\eta)$ be the angle between the normal to a plane curve and a fixed direction in the plane. Then, as is well known, the angular increment between two points on the curve is given by:

$$\chi_2 - \chi_1 = R_0 \int_{\eta_1}^{\eta_2} \frac{d(\eta)}{R(\eta)}.$$

Hence, an angular discontinuity of $(\Delta\chi)$ radians is equivalent to a curvature concentration of

$$\frac{\Delta\chi}{R_0}\delta(\eta = \eta_0)$$

where $\delta(\eta = \eta_0)$ is the Dirac delta function.

Therefore, given a curve with curvature $[1/R(\eta)]'$ and given in addition that the curve has a finite number of angular discontinuities $(\Delta\chi)_i$, then the modified expression for its curvature becomes:

$$\frac{1}{R(\eta)} = \left[\frac{1}{R(\eta)}\right]' + \frac{1}{R_0}\sum_i (\Delta\chi)_i \delta(\eta = \eta_i).$$

As an example, take the circumference of a cylindrical shell under internal pressure p as two circular arcs of radius r and angular jumps (between the two arcs) of magnitude β (Fig. 1). Here:

$$R_0 = r(1-\beta/\pi)$$

$$\varepsilon = \frac{\beta}{\pi - \beta}$$

$$\frac{1}{R(\eta)} = \frac{1}{r} + \beta/R_0\left[\delta(\pi/2) + \delta\left(\frac{3\pi}{2}\right)\right].$$

Expanding $\frac{1}{R(\eta)}$ as a Fourier cosine series, one obtains:

$$\frac{R_0}{R(\eta)} = 1 + \frac{2\beta}{\pi}\sum_{2,4,6}^\infty (-1)^{n/2} \cos n\eta.$$

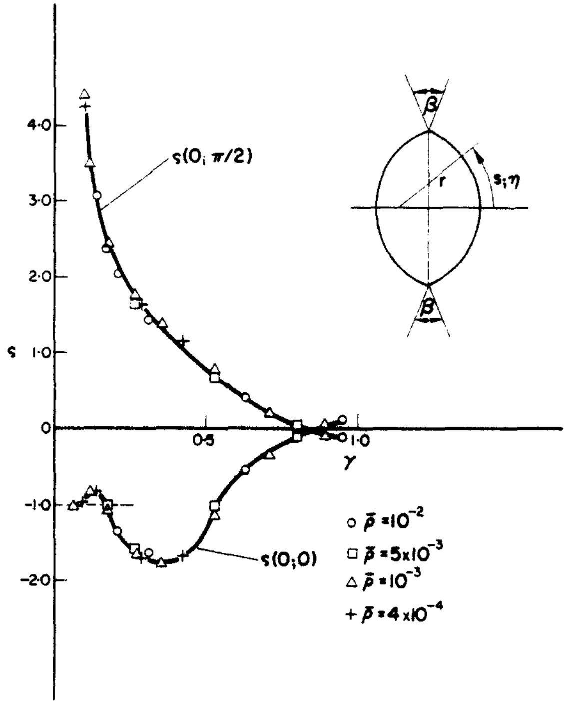

FIG. 1. Two-circular-arc shell.

Note that the coefficients of the series are bounded. Therefore, although the series for $1/R(\eta)$ is not convergent, the series for the solution of the shell problem does converge in the interior of the shell $|\xi| < 1$, giving rise to a *smooth surface*. The expression for the circumferential curvature of the deformed shell is:

$$\kappa_s = 1+\frac{2\beta}{\pi}\sum_{2,4,6}^{\infty}(-1)^{n/2}\mu_n''(\xi)\cos n\eta. \tag{73}$$

The solution presented here loses part of its accuracy near $\eta = \pi/2, 3\pi/2$. This is due to the fact that linearization has been performed on locally large changes in curvature. The effects on the geometry are not too important since the latter is related to the averaging of concentrations, but the accuracy of the expressions for the stresses (in particular N_s) may be significantly reduced. A more satisfactory approximation for N_s may be obtained by observing that if the first two terms are omitted in (40) as higher order terms, then the expression for N_s would be given by

$$N_s = \bar{p}/\kappa_s.$$

The last equation may be used to calculate N_s after κ_s has been obtained from (73). The alternative is to use more perturbation terms.

It must be noted here that the solution corresponds to the case of a pure membrane only, which cannot accept moments. In a real shell, physical considerations demand that moments exist along the crease lines of the shell and thus the solution will be modified.

However, as the pressure increases, a plastic "hinge" is formed along the crease until the latter is obliterated and the solution (73) should approach the physical situation for such shells too.

SOME NUMERICAL RESULTS

(a) *Two circular-arc-shell*

Using equation (73) and the expression for R_0, the circumferential curvature of the deformed shell may be written in the form:

$$b_{ss} = \frac{1}{r}\frac{\pi+\beta\zeta(\xi,\eta)}{\pi-\beta}$$

where

$$\zeta(\xi,\eta) = 2\sum_{n=2,4,6}^{\infty}(-1)^{n/2}\mu_n''(\xi)\cos n\eta.$$

The values of $\zeta(0, 0)$ and $\zeta(0, \pi/2)$ (along the minor and major axes at the shell centre) are plotted in Fig. 1 against the shell parameter γ [equation (72)] for a range of values:

$$\tfrac{1}{3} \leq \alpha \leq 3; \qquad 4\times 10^{-4} \leq \bar{p} \leq 10^{-2}.$$

It is seen from the graph that a single smooth curve can be drawn through the points with very minor deviations. It thus appears that within this range of values γ is the significant parameter which determines the shell behaviour.

(b) *Simple oval*

The equation of the oval is given in the form

$$R(\eta) = \bar{R}_0 + R_2\cos 2\eta$$

The radius of its isoperimetric circle is

$$R_0 = [\bar{R}_0^2 - R_2^2]^{\frac{1}{2}}.$$

Numerical results are given for the following quantities:

$$n^{ss}(\xi = 0) = p\{[\bar{R}_0^2 - [1-\mu_2''(0)^2]R_2^2]^{\frac{1}{2}} + R_2\mu_2''(0)\cos 2\eta\}$$

$$n^{xx}(\xi = 0) = -\frac{4pR_2}{\alpha^2}\mu_2(0)\cos 2\eta$$

$$n^{xs}(\xi = 1) = \frac{2pR_2}{\alpha}\mu_2'(1)\sin 2\eta.$$

The quantities $\mu_2''(0)$, $\mu_2'(1)$ and $\mu_2(0)$ are plotted in Fig. 2 against the shell parameter γ for the same range of values as in the previous example. As before, a single smooth curve can be drawn through the points with minor deviations, so that γ appears to be the significant parameter which determines the shell behaviour. It appears from Fig. 2 that for $\gamma < 0{\cdot}2$ linear membrane theory gives good results. For high values of γ, nonlinear effects become significant and soon dominate the solution.

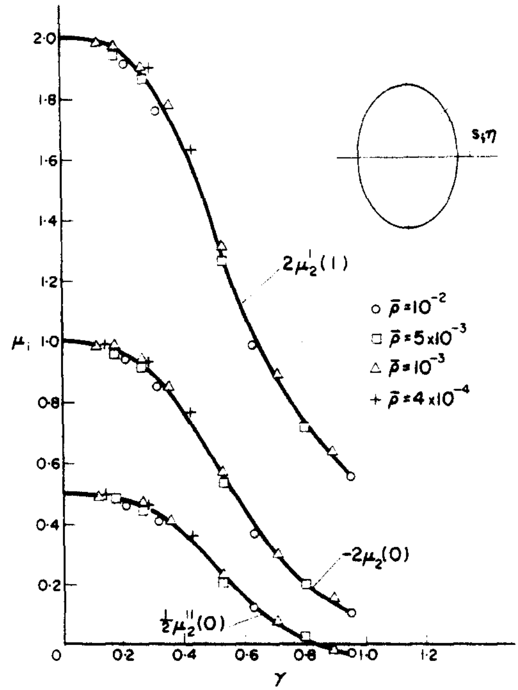

FIG. 2. Simple oval shell.

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Абстракт—Даётся нелинейная зацача мембраны рри помощи выравнений на кривизны и напряжения /или деформации/, в качестве переменных поля. Исследуются приближенные граничные условия, в смысле части изложения теории.

Обсуждаются важные характеристические свойсмва уравнений. К ним принадлежат следуюшие темы: зависимостъ от начплъной формы, типы уравнений, возможностъ выяснения аномалий линейной задачи мембраны, устойчивостъ и задача краевого эффекта мембраны. Даётся также схема рядов возмущения для решения уравнений.

Используемостъ предлагаемой теории для решения практических задач указывается путём определения решения, касающегося задачи некруглой цилиндрической мембранной оболочки под действием горизонтальной нагрузки.